# Information Theory Meets Expected Utility: The Entropic Roots of Probability Weighting Functions

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## Introduction

- The most widely applied **non-expected utility theories**[1,2,3] combine the classical core of expected utility theory[4] with a **concave utility function** and **non-linear probability weighting** to account for findings unexplained by EUT[5,6,7]

$$EV(X) = \sum_{i=1}^{n} \boldsymbol{\pi(p_i)} * \boldsymbol{v(x_i)}$$

- Despite of decades of research, the **psychological processes** underlying the estimated shape of these functions remain unclear
- Earlier work has suggested that **concave utility** could reflect **decision by sampling**[8], **efficient coding**[9, 10, 11], **heuristic processing**[12] or **attentional processes**[13, 14], whereas **probability weighting** has been suggested to result from **sensitivity to extreme probabilities**[1, 15], a **log-odds representation** of frequency and probability-related information[16], and/or **bounded rationality**[17, 18]
- Based on existing work in perceptual and cognitive psychology[19, 20, 21, 22] and neuroscience[23, 24], the present approach builds on the assumption that **a reduction in uncertainty carries utility**

## Valence-Weighted Distance (VWD) [25]

- The **perception of a probability** $p$ is influenced by the **amount of uncertainty reduction it carries**
  - relative to the **uniform** distribution (maximum entropy)
  - depending on **other probabilities** in the distribution (actual entropy)
- Formally:

$$VWD(p) = \frac{p^{\left(1-\frac{H_{dist}}{H_{max}}\right)}}{\sum_{i=1}^{n} p_i^{\left(1-\frac{H_{dist}}{H_{max}}\right)}}$$

where $Hdist = -\sum p_i \log_2(p_i)$[26], $Hmax = \log_2(n)$, and $n$ denotes the number of probabilities in the distribution that $p$ is embedded in

- This yields an $n$-dimensional function that has its fixed point at $1/n$ and curvature that depends on the entropy of the distribution

**Note:** Existing probability weighting functions cannot reflect changes in $n$ or distributional shape without changes in fitted parameters

**Figure 1.** Existing probability weighting functions[3, 27, 28, 29, 30]

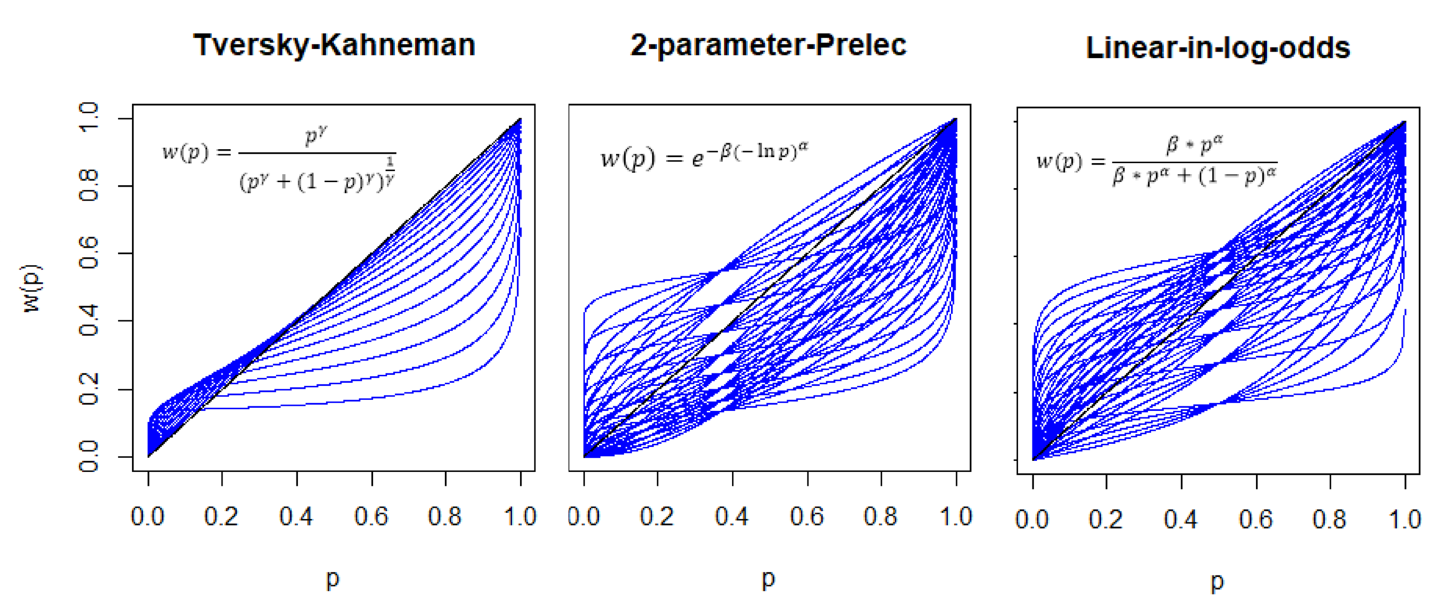



**Figure 2.** $VWD(p)$ when $n$ is varied from to 2 to 7 while retaining maximum entropy (left) and when entropy is varied while $n = 3$ (right)

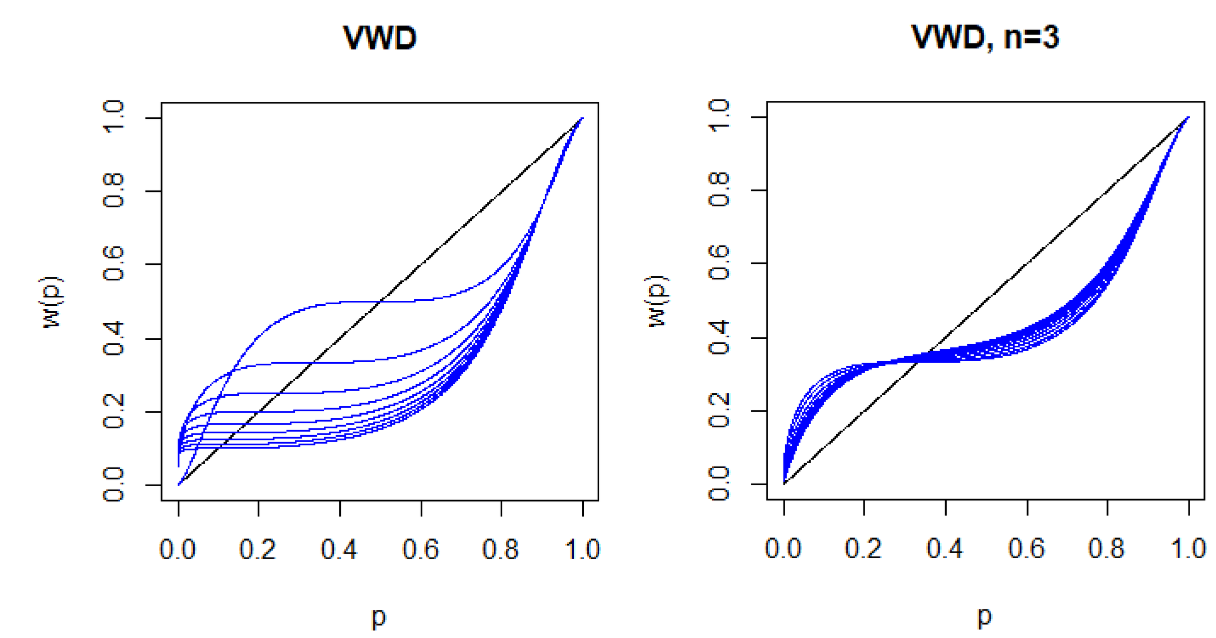
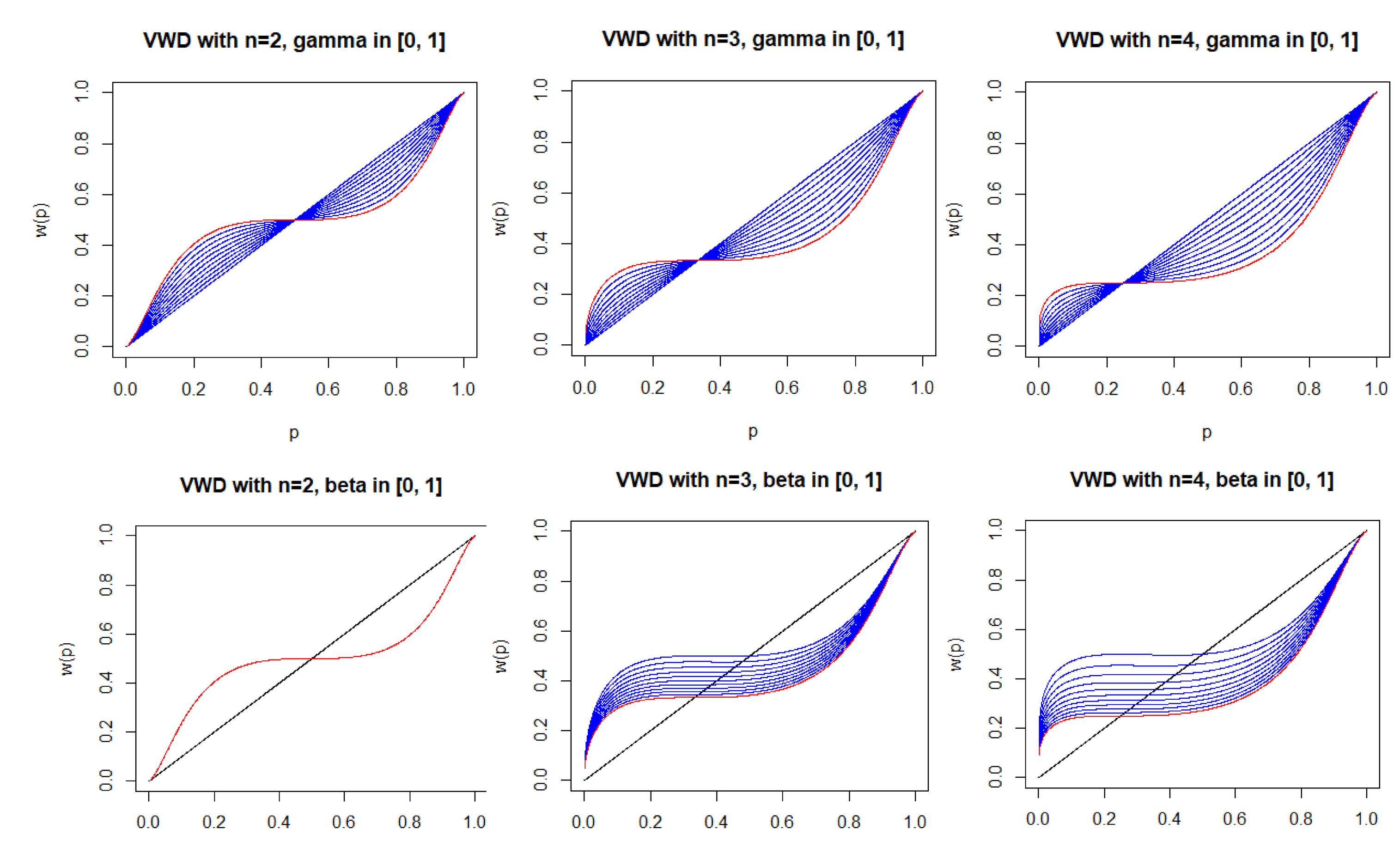
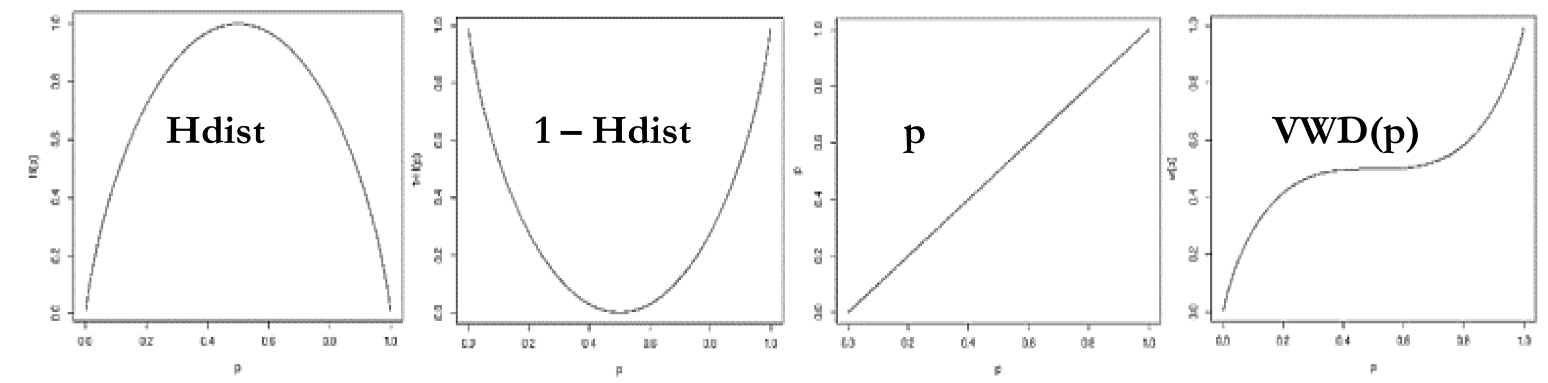



**Figure 3.** Parameterized $VWD(p)$ when γ is varied from 0 to 1 while fixing β to 1 (upper row) and when β is varied from to 0 to 1 while fixing γ to 1 (lower row) for $n = 2$ (left), $n = 3$ (middle), and $n = 4$ (right).




## Illustration with $n$ = 2

Utility of uncertainty reduction merged with utility of increased likelihood of gain




$$VWD(p) = \frac{p^{(1-Hdist)}}{p^{(1-Hdist)} + (1-p)^{(1-Hdist)}}$$

For $n = 2$, $Hmax = 1$

Compare to linear-in-log-odds: $w(p) = \frac{\beta * p^{\alpha}}{\beta * p^{\alpha} + (1-p)^{\alpha}}$

## Parameterized VWD

$$VWD(p_1) = \frac{p_1^{\left(1-\gamma*\frac{H_{dist}}{H_{max}}\right)}}{p_1^{\left(1-\gamma*\frac{H_{dist}}{H_{max}}\right)} + \beta * \sum_{i=2}^{n} p_i^{\left(1-\gamma*\frac{H_{dist}}{H_{max}}\right)} + (1-\beta)*(1-p_1)^{\left(1-\gamma*\frac{H_{dist}}{H_{max}}\right)}}$$

where β, γ ϵ [0, 1] reflect attention given to $n$ and entropy

## Contributions

**1. Probability weighting**

- Explains probability weighting with **a simple principle**
- Takes **context of** $\boldsymbol{p}$ ($n$, shape of distribution) into account
- Makes **novel, empirically testable predictions**
  - Location of fixed point determined by $n$
  - Curvature determined by shape of distribution

**2. Expected utility theory and information theory**

- Captures the psychological impact of **outcome-probability associations**[31] cf. [32, 33]

3. **Log-odds representation of frequency and probability**

- Provides a parsimonious explanation for the $1/n$ puzzle[16]

$$log\frac{VWD(p_i)}{1-VWD(p_i)} = log\frac{p_i^{\left(1-\frac{H_{dist}}{H_{max}}\right)}}{\sum_{j=1}^{n} p_j^{\left(1-\frac{H_{dist}}{H_{max}}\right)}} \quad \text{where } i \neq j$$

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